PythonStochasticDiffEq

October 13, 2019

1 Simple Stochastic Differential Equation Model in Python

Let Z_t for $t \in [0, \infty)$ be the stochastic process for which:

1. $\Delta_{t_0}^{t_1} Z = Z_{t_1} - Z_{t_0}$ is normally distributed with mean 0 and variance $t_1 - t_0 > 0$,
2. $\Delta_{t_0}^{t_1} Z$ and $\Delta_{s_0}^{s_1} Z$ are independent random variables whenever $[t_0, t_1]$ and $[s_0, s_1]$ are disjoint intervals.

In essence, the infinitesimal change in the random variable, dZ_t, behaves like a normally distributed random variable with mean 0 and variance "dt".

A stochastic differential equation can be expressed as

$dX_t = r(X_t, t)dt + \sigma(X_t, t)dZ_t$

with initial value $X_0 = c$, a constant random variable, and r, σ functions of X_t and t.

If we let $r(X_t, t) = r_0 X_t$ for $r \in \mathbb{R}$ and $\sigma(X_t, t) = \sigma_0 X_t$ with $\sigma_0 > 1$, we are searching for the stochastic process X_t for which

$dX_t = X_t(r_0 dt + \sigma_0 dZ_t)$

Let us model the realization of a path $t \mapsto X_t(\omega)$, where ω is in the the event space Ω.

We make use of the following facts:

1. If $\Delta_{t_0}^{t_1} Z$ is normally distributed with mean 0, variance $t_1 - t_0$, then $\Delta_{t_0}^{t_1} Z = \sqrt{t_1 - t_0} Y$, where Y is a standard, normal random variable.
2. Consider the sub-division of the interval $[0, T]$ by intervals of length T/N. Let $t_k = kT/N$, and recursively define for $N \geq k \geq 1$:

$$X_{t_k} - X_{t_{k-1}} = X_{t_{k-1}}(r_0(t_k - t_{k-1}) + \sigma_0 \Delta_{t_{k-1}}^{t_k} Z)$$

.

Then for t_k,

$$X_{t_k} = X_{t_{k-1}}(1 + r_0 \frac{T}{N} + \sigma_0 \sqrt{T/N}\, Y)$$

.

i.e. at time t_k, the random variable is determined by what ocurred at time t_{k-1}. So to model a path or realization, we can plot t_k against X_{t_k}. To approximate our random variable X_T, we sum over the values $X_{t_k} - X_{t_{k-1}}$:

$X_T \approx \sum_{k=1}^{N} X_{t_k} - X_{t_{k-1}}$ for N large.

Importing some of the things we need:

```
In [100]: import random
          import math
          import matplotlib.pyplot as plt
          plt.rcParams["figure.figsize"] = (8,6)
          import numpy as np
```

Setting up our parameters:

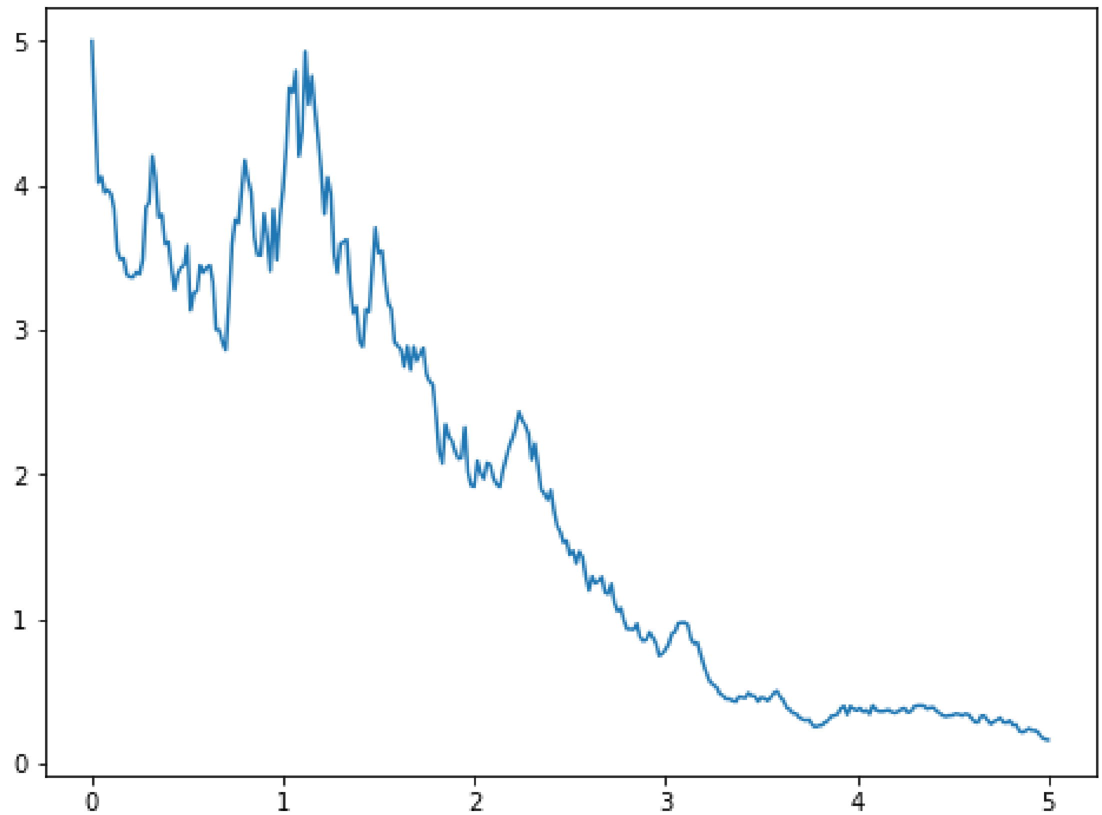
```
In [68]: X_0=5
         r_0=-0.001
         sigma_0=0.45
         T=5
         N=300
```

Setting up our t_k points:

```
In [69]: t=[]
         for k in range(0,N+1):
             t+=[T*k/N]
```

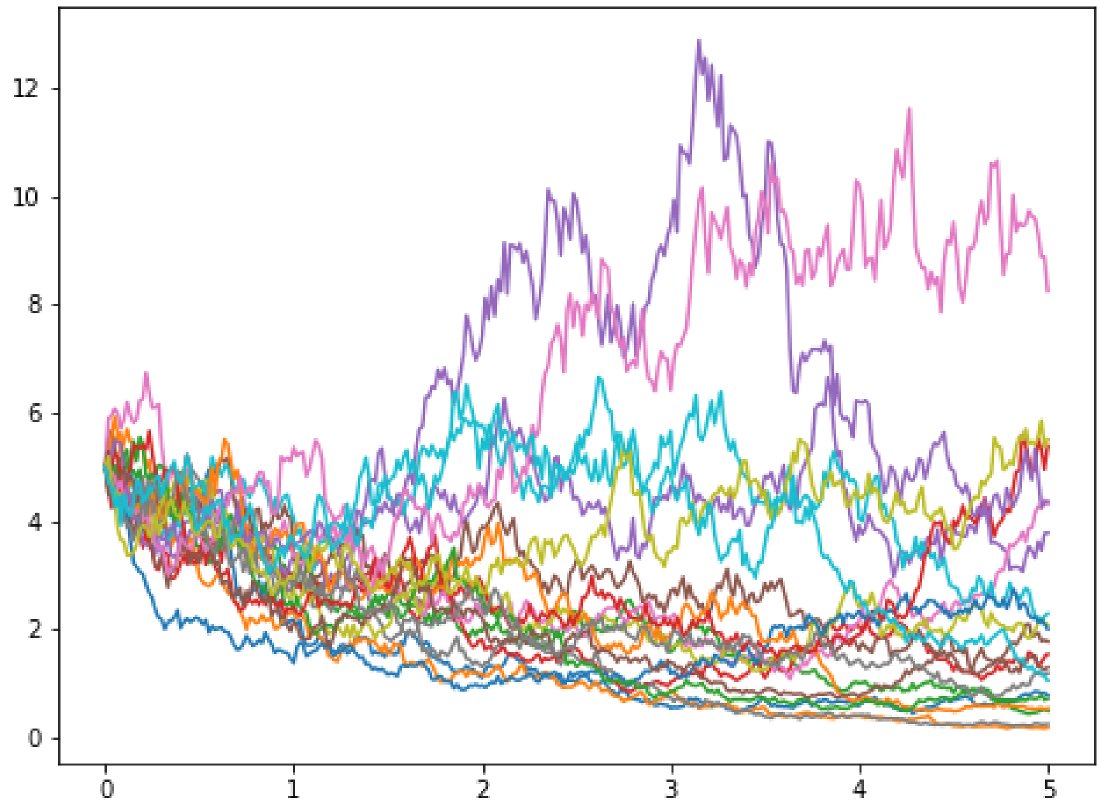
Every time the following code is run, a new realization will be graphed:

```
In [94]: x=[X_0]
         x+=[x[0]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1))]
         for k in range(2,N+1):
             x+=[x[k-1]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1))]
         plt.plot(t,x)
         plt.show()
```



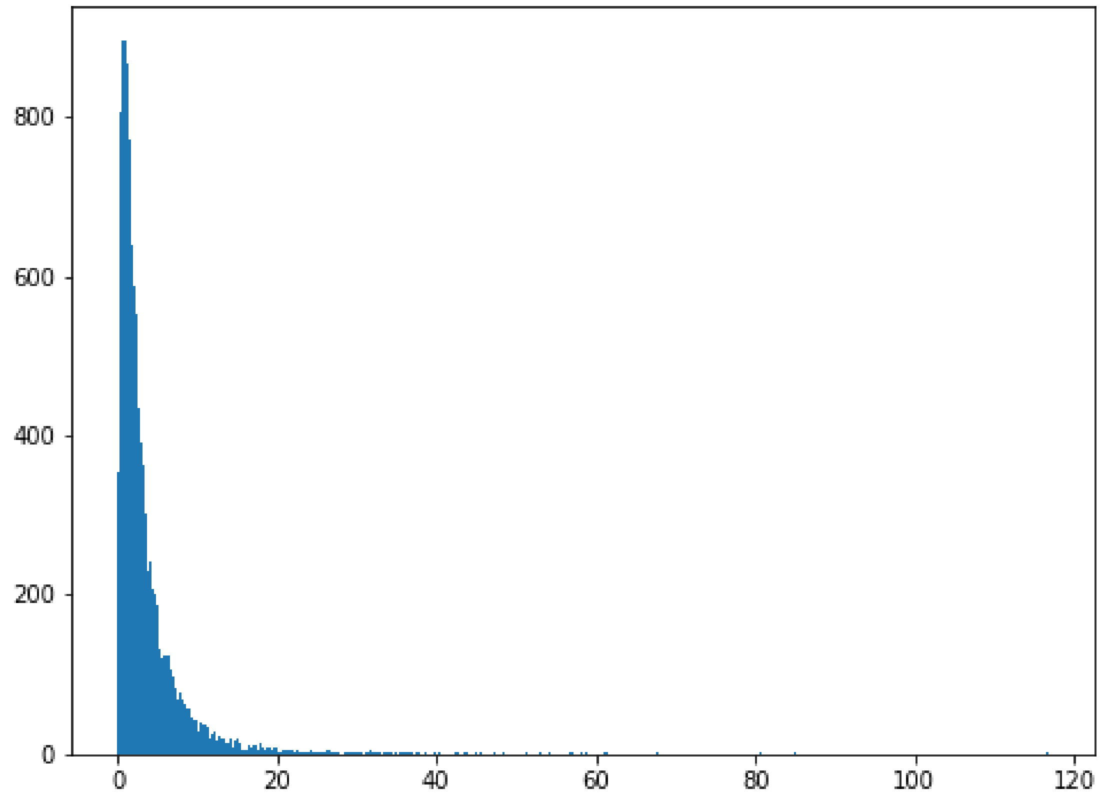
Now let's look at about 20 trials:

```
In [96]: for j in range(0,20):
             x=[X_0]
             x+=[x[0]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1))]
             for k in range(2,N+1):
                 x+=[x[k-1]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1)
             plt.plot(t,x)
         plt.show()
```



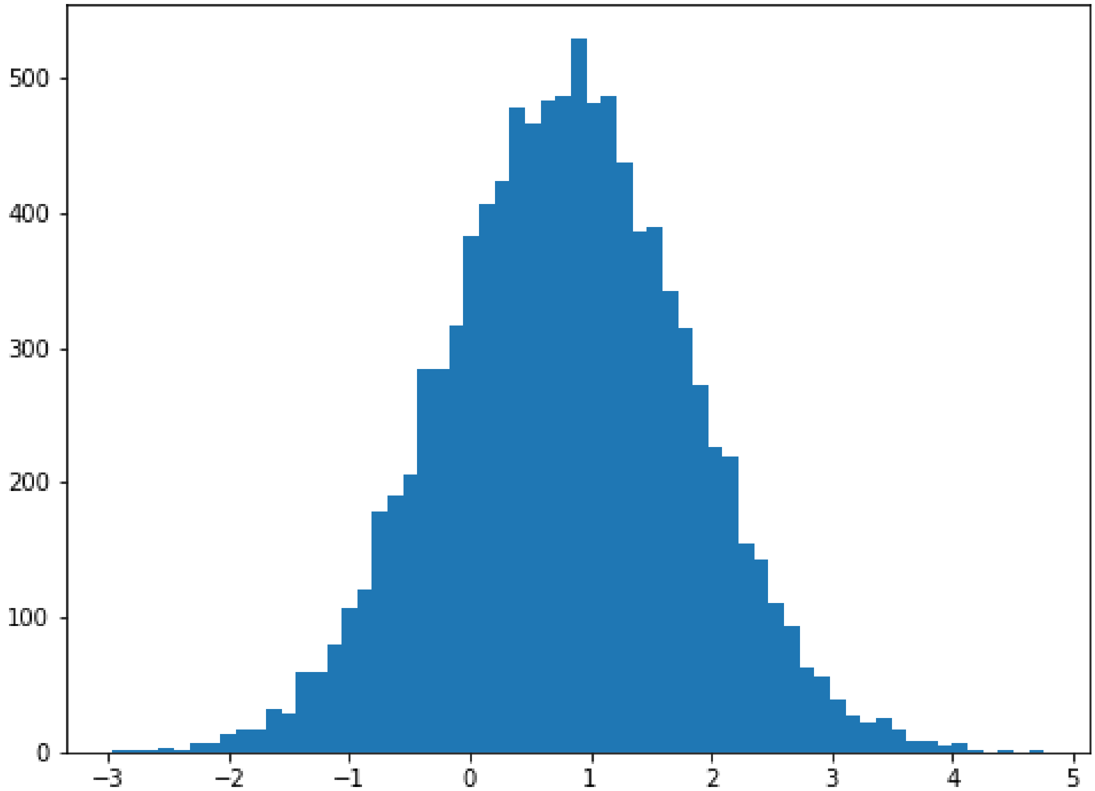
To determine a trend, we will store 10,000 realizations as the value $x_T(\omega)$, and then plot a histogram.

```
In [111]: X_T=[]
          for j in range(0,10000):
              x=[X_0]
              x+=[x[0]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1))]
              for k in range(2,N+1):
                  x+=[x[k-1]*(1+r_0+sigma_0*math.sqrt(T/N)*random.normalvariate(0,1
              X_T+=[x[N]]
          plt.hist(X_T,bins='auto');
```



Which looks a lot like a log normal distribution. . . Just to verify our suspicions:

```
In [112]: L_T=[]
          for x in X_T:
              L_T+=[math.log(x)]
          plt.hist(L_T,bins='auto');
```




```
In [ ]:
```